of normal sculpture. As far as I have seen and read, the range is unique in this systematic tripartite arrangement of normally and glacially sculptured forms. A fuller account of the range will be prepared for the *Bulletin of the American Geographical Society*.

DEFINITION OF LIMIT IN GENERAL INTEGRAL ANALYSIS

By Eliakim Hastings Moore

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

Presented to the Academy, November 8, 1915

1. *General Analysis.* The problem of Science is the organization of and the study of the interrelations amongst the objects and phenomena of Nature. Analogous objects or phenomena are grouped into classes. In the progress of Science, with the discovery of new objects or phenomena or interrelations, the bases of classification initially of necessity superficial become more fundamental; thus in Physics, Electricity and Magnetism and later Light merge in Electromagnetism.

Mathematics with its source in Nature progresses in similar fashion. Hence, remembering that the objects or phenomena of Mathematics may be theories (doctrines), we may enunciate the following heuristic principle:

> The existence of analogies between central features of various theories implies the existence of a more fundamental general theory embracing the special theories as particular instances and unifying them as to those central features.

After the development of such a general theory, the fact that the various theories are instances of the general theory implies as an obvious consequence (and accordingly eclipses in importance) the analogies between the central features of the various theories. In illustration of the heuristic principle may be adduced the theories of General Analysis mentioned below.

Analysis is the branch of Mathematics devoted to the classification of and the study of the interrelations amongst numerically valued functions. A (single-valued) function τ or $\tau(p)$ is a table (or rule or process) assigning to every element or member p of a certain class or range $\mathfrak{P}$ a definite element q of a certain class $\mathfrak{Q}$. It is numerically valued in case the functional values $q, = \tau(p)$, are numbers real or complex. Although not always numerical, the independent variable p of a function τ considered in a theory of Classical Analysis is always of specified nature; e.g., the variable p may be a curve or a numerically valued function, $\tau(p)$ being the length of the curve p or the average value over its range of definition of the function p. However, the nature of the element p is not fully specified in Fréchet's theory[1] (1906) of sets of elements p of a class $\mathfrak{P}$ and of continuous functions on such sets. This general theory has as instances the theories for the linear continuum $\mathfrak{P}$ and for the n-dimensional space $\mathfrak{P}$ initiated by G. Cantor and for the class $\mathfrak{P}$ of continuous curves due especially to Arzelà. Fréchet conditions his class $\mathfrak{P}$ by a definite but undefined relation L (the concept of convergence of a sequence $\{p_n\} : p_1, p_2, \ldots, p_n, \ldots$, of elements p to an element p_0 of $\mathfrak{P}$ as a limit) possessing certain properties; these properties of the relation L are in the special theories immediate consequences of the current definitions of the relation L in those theories.

Those theories of Analysis in which at least certain of the functions involved are on a range $\mathfrak{P}$ of elements p of a nature not specified, or at least not fully specified, we may designate as theories of General Analysis. Thus, Fréchet's theory with basis ($\mathfrak{P}$; L) is a theory of General Analysis, as is likewise my theory[2] (1910) of classes of functions on a general range $\mathfrak{P}$.

A general range $\mathfrak{P}$ is an arbitrary particular range $\mathfrak{P}$ with abstraction of its particular features, e.g., Fréchet's range $\mathfrak{P}$ with abstraction of the feature L. Properties of functions, classes of functions, etc., on or connected with a general range $\mathfrak{P}$ (whose definitions accordingly involve no particular features of the range $\mathfrak{P}$) are of 'general reference,' while others are of 'special reference.' Thus, relative to a linear interval $\mathfrak{P}$, the continuity of a function is of special reference, while the property of uniformity of convergence of a sequence of functions and the property of the class of all continuous functions that the limit of a uniformly convergent sequence of the class belongs to the class are properties of general reference.

2. *General Integral Analysis.* A theory of General Analysis involving a numerically valued single-valued functional operation J (of the type of definite integration) on a class $\mathfrak{K}$ of functions κ whose range of definition involves a general range $\mathfrak{P}$ we may designate as a theory of General Integral Analysis. Thus, my theory[3] of linear integral equations is a theory of General Integral Analysis. This general theory has as instances the classical theory due to Fredholm and Hilbert-Schmidt for the case of continuous functions, and other classical theories.

3. *Definition of Limit in General Integral Analysis.* In a subsequent note I shall indicate a general theory of linear integral equations having as instance Hilbert's theory of functions of infinitely many variables.

The integration process J, undefined in my earlier theory, is in this theory defined. Its definition turns on the definition of limit which I wish to explain in this note.

By way of example consider on the range $\mathfrak{P}^{\text{III}}$ ($p = 1, 2, 3, \ldots, n, \ldots$) a numerically valued function $\alpha(p)$ with absolutely convergent sum $J\alpha, = \alpha(1) + \alpha(2) + \alpha(3) + \ldots$ $J\alpha$ is the limit in the classical sense, as n increases without bound, of $J_n\alpha, = \alpha(1) + \alpha(2) + \ldots + \alpha(n)$. Here the definitions of $J_n\alpha$ and of limit are of special reference. However, taking (not the first n but) any finite set σ say of n elements p: $p_1 < p_2 < \ldots < p_n$, of the range $\mathfrak{P}^{\text{III}}$, we secure definitions of $J_\sigma\alpha, = \alpha(p_1) + \alpha(p_2) + \ldots + \alpha(p_n)$, and of limit which are of general reference.

Indeed, consider at once a general class $\mathfrak{P}$ and a numerically valued (possibly many-valued) function F on the class $\mathfrak{S}$ of all finite sets σ of elements p of the range $\mathfrak{P}$. (In the example cited $F(\sigma) = J_\sigma\alpha$). We say that the number a is the limit as to σ of the function $F(\sigma)$, or that, as to σ, $F(\sigma)$ converges to a, in notation:

$$\underset{\sigma}{\mathrm{L}}\ F(\sigma) = a, \tag{1}$$

in case for every positive number e there exists a set σ_e (depending on e) of such a nature that for every set σ including σ_e $|F(\sigma) - a| \leqq e$, in symbols:

$$e : \supset : \exists\, \sigma_e \ni \sigma^{\supset \sigma_e} . \supset . |F(\sigma) - a| \leqq e. \tag{2}$$

If for a set σ including σ_e $F(\sigma)$ is many-valued the understanding is that the final inequality holds for every value of $F(\sigma)$. (The notation e denotes a positive number; the notations n, m used below denote positive integers.)

The L_σ of (1) is a single-valued functional operation of the type of a definite integral, in that it reduces every function $F(\sigma)$ of the class of all functions convergent as to σ to a number a. Accordingly, the class $\mathfrak{P}$ being general, this definition of limit (even apart from its use in the theory mentioned) belongs to General Integral Analysis.

In order to obtain definitions of various modes of convergence in case the function F involves a parameter we notice the equivalent forms (3, 4, 5) of the definition.

$$n : \supset : \exists\, \sigma_n \ni \sigma^{\supset \sigma_n} . \supset . |F(\sigma) - a| \leqq 1/n, \tag{3}$$

$$\exists \{\sigma_m\} \ni n : \supset : \sigma^{\supset \sigma_n} . \supset . |F(\sigma) - a| \leqq 1/n. \tag{4}$$

viz., there exists a sequence $\{\sigma_m\} : \sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$, of sets σ of such a nature that for every n and set σ including σ_n $|F(\sigma) - a| \leqq 1/n$.

$$\exists \{\sigma_m\} \cdot \ni n : \supset : \exists m_n \ni \sigma^{\supset \sigma_{m_n}} . \supset . |F(\sigma) - a| \leqq 1/n, \qquad (5)$$

viz., there exists a sequence $\{\sigma_m\}$ such that for every positive integer n there exists a positive integer m_n (depending on n) such that for every set σ including σ_{m_n} $|F(\sigma) - a| \leqq 1/n$.

Now let the function F involve a parameter u on a range $\mathfrak{U}$ and suppose that $F(\sigma, u)$ converges as to σ for every u of $\mathfrak{U}$; the limit is a single-valued function, say φ, of u, in notation:

$$\underset{\sigma}{L}\, F(\sigma, u) = \varphi(u) \qquad (u), \qquad (6)$$

that is,

$$u . : \supset : . \exists \{\sigma_{um}\} \ni n : \supset : \exists m_{un} \ni \sigma^{\supset \sigma_{um_{un}}} . \supset . |F(\sigma, u) - \varphi(u)| \leqq 1/n, \quad (7)$$

viz., for every u of $\mathfrak{U}$ there exists a sequence $\{\sigma_{um}\}$ of sets σ (depending on u) such that for every positive integer n there exists a positive integer m_{un} (depending on u and n) such that for every σ including $\sigma_{um_{un}}$ $|F(\sigma, u) - \varphi(u)| \leqq 1/n$.

The convergence is semiuniform over the range $\mathfrak{U}$ in case a single sequence $\{\sigma_m\}$ is effective as the sequence $\{\sigma_{un}\}$ for every u of $\mathfrak{U}$, and it is uniform in case moreover for every n a single positive integer m_n is effective as the positive integer m_{un} for every u of $\mathfrak{U}$, that is, the notations:

$$\underset{\sigma}{L}\, F(\sigma, u) = \varphi(u) \qquad (u;\ \text{semiunif.}); \qquad (8)$$

$$\underset{\sigma}{L}\, F(\sigma, u) = \varphi(u) \qquad (u;\ \text{unif.}), \qquad (9)$$

have the respective meanings:

$$\exists \{\sigma_m\} \ni u . : \supset : . n : \supset : \exists m_{un} \ni \sigma^{\supset \sigma_{m_{un}}} . \supset . |F(\sigma, u) - \varphi(u)| \leqq 1/n; \quad (10)$$

$$\exists \{\sigma_m\} \ni n . : \supset : . \exists m_n \ni u : \supset : \sigma^{\supset \sigma_{m_n}} . \supset . |F(\sigma, u) - \varphi(u)| \leqq 1/n. \quad (11)$$

If v is a numerically valued single-valued function of u on $\mathfrak{U}$, we define semiuniformity and uniformity of convergence relative to v as scale function over the range $\mathfrak{U}$, in notation: as in (8, 9) with the parentheses replaced by (u; semiunif. $v(u)$), (u; unif. $v(u)$) respectively, by replacing in the definitions (10, 11) the final $1/n$ by $|v(u)|/n$. Thus, semiuniformity and uniformity are absolute, i.e., relative to the scale function $v(u) = 1$.

The definitions of uniformity (absolute and relative) may be simplified by omitting '$\exists\ m_n \ni$' and replacing 'σ_{m_n}' by 'σ_n'. This form of definition is suggested directly by (4); the more complicated form (5) with its redundant existential feature m_n serves however to suggest the definitions of semiuniformity (absolute and relative), and relative semiuniformity proves to be of importance in the applications.

[1] M. Fréchet, Sur quelques points du calcul fonctionnel, *Palermo, Rend. Circ. mat.*, **22**, 1–74 (1906).

[2] E. H. Moore, Introduction to a Form of General Analysis, 1–150, *The New Haven Mathematical Colloquium*, Yale Univ. Press, 1910. Cf. also, E. H. Moore, On a Form of General Analysis with Application to Linear Differential and Integral Equations, *Atti* IV *Cong. Inter. Mat.* (Roma, 1908), **2**, 98–114 (1909).

[3] E. H. Moore, On the Foundations of the Theory of Linear Integral Equations, *Bull. Amer. Math. Soc.*, Ser. 2, **18**, 334–362, (1912). On the Fundamental Functional Operation of a General Theory of Linear Integral Equations, *Proc. Fifth Inter. Congr. Math.* (Cambridge, Aug., 1912) **1**, 230–255 (1913).

NOTICE OF SCIENTIFIC MEMOIR

The Variations and Ecological Distribution of the Snails of the Genus Io. By Charles C. Adams, New York State College of Forestry, Syracuse, N. Y. Second Memoir of Volume 12 of the Memoirs of the National Academy of Sciences, Washington, 1915. 1–184 p., 64 pl.

Io is a large gilled snail which lives only in the Tennessee River system. It is extremely variable, shows a remarkable distribution in the streams and in this it appears to be related to the physical history of the drainage. Throughout this Memoir emphasis is placed upon relating the changes of the animals to the changes in the environment. The general natural history of the snails is briefly summarized, the local races are described and the localities from which the collections studied were secured is given in detail. The shells were grouped in convenient classes for descriptive purposes. The diameter of the shell, its degree of globosity, and the degree of development of the spines were determined quantitatively. These qualities are discussed by streams and drainage systems. In the parallel flowing Powell, Clinch and (North Fork) Holston rivers, the shells are smooth or with low spines in the headwaters, and down stream have longer spines. This condition is quite remarkable and no previous detailed investigation has been made of a problem of this character.

The development or evolution of the gross environment is discussed. The author states "It is considered that a knowledge of the development and structure of the environment is as essential a part of the problem as is the development and structure of the animals themselves." An outline history of the Tennessee drainage is given. In the past this family of shells was thought to have originated in the Northwest (Laramie) but the author suggests an alternative hypothesis, that they originated in the southeast.